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Essential elements in connected k-polymatroids

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ABSTRACT

It is a well-known result of Tutte that, for every element x of a connected matroid M, at least one of the deletion and contraction of x from M is connected. This paper shows that, in a connected k-polymatroid, only two such elements are guaranteed. We show that this bound is sharp and characterize those 2-polymatroids that achieve this minimum. To this end, we define and make use of a generalized parallel connection for k-polymatroids that allows connecting across elements of different ranks. This study of essential elements gives results crucial to finding the unavoidable minors of connected 2-polymatroids, which will appear elsewhere.

1. Introduction

A classical result of Tutte is that, for every element x of a connected matroid M, either $M\backslash x$ or M/x is connected. This property of being able to either delete or contract any element while maintaining connectivity, however, does not hold for k-polymatroids. We call an element x of a connected k-polymatroid *essential* if both its deletion and contraction from the k-polymatroid destroy connectivity. In this paper, we show that every k-polymatroid has at least two elements that are non-essential, show that this bound is sharp for each integer k exceeding one, and characterize all 2-polymatroids with exactly two non-essential elements.

Additional motivation for this paper comes from the desire to find the unavoidable minors for connected 2-polymatroids, which is done in [2]. This study of essential elements turns out to be a crucial step in that endeavor. In fact, one may divide the class of unavoidable minors for connected 2-polymatroids into two categories: those that resemble circuits and cocircuits in matroids, and those that have exactly two non-essential elements.

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The main results, Theorems 4.3 and 4.9, are stated and proved in Section 4. The concepts of 2-sum and parallel connection for k-polymatroids, ideas that play an important role in the proofs of the main results, are studied in Section 3. Polymatroid-theoretic preliminaries are given in Section 2.

2. Polymatroids

Let M be a matroid with ground set E and rank function r. The pair (E, r) is an example of a 1-polymatroid. In fact, the class of 1-polymatroids is exactly the class of matroids. For an arbitrary positive integer k, we now define a k-polymatroid noting that it is very much like a matroid but allows individual elements to have ranks up to k.

Much of the polymatroid-theoretic language in this paper follows [7]. Let E be a finite set and f be a function from the power set of E into the integers. We say that f is *normalized* if $f(\emptyset) = 0$; f is *submodular* if $f(X) + f(Y) \geqslant f(X \cup Y) + f(X \cap Y)$ for all $X, Y \subseteq E$; and f is *increasing* if $f(X) \leqslant f(Y)$ whenever $X \subseteq Y \subseteq E$. We call the pair (E, f) a *polymatroid* $\mathcal{Q}$ if f is normalized, submodular, and increasing. The set E is called the *ground set* of $\mathcal{Q}$ while f is the *rank function*. For a positive integer k, a polymatroid (E, f) is a *k-polymatroid* if $f(z) \leqslant k$ for all z in E. For ease of notation, a rank-1 element of a k-polymatroid is called a *point*; a rank-0 element of a k-polymatroid is called a *loop*. If a and b are elements of a k-polymatroid such that $f(\{a, b\}) = f(a) = f(b)$, then we say that a and b are *parallel*.

Examples

An important way to obtain a k-polymatroid from a matroid is as follows. Given a matroid M with ground set S and rank function r, we obtain a k-polymatroid $\mathcal{Q} = (E, f)$ by taking E to be some subset of the set of flats of M of rank at most k and letting $f(X) = r(\bigcup_{x\in X} x)$ for all subsets X of E. Indeed, every k-polymatroid can be obtained in this way (see, for example, [3,5]). This fundamental fact allows us, in particular, to think of 2-polymatroids as an arrangement of loops, points, and lines of a matroid.

Another natural class of 2-polymatroids arises from graphs. To see this, let G be a graph and set $E = E(G)$. For a subset X of E, define a function f by $f(X) = |V(X)|$ where $V(X)$ is the set of vertices of G that meet some edge of X. Then (E, f) is a 2-polymatroid. We will call the 2-polymatroids that can be represented in this way *Boolean* and note that there is a one-to-one correspondence between the class of Boolean 2-polymatroids and the class of graphs without isolated vertices [8].

Finally, we consider k-polymatroids that are derived from other polymatroids. Let $\mathcal{Q}_1 = (E, f_1)$ and $\mathcal{Q}_2 = (E, f_2)$ be k-polymatroids on the same ground set. It is not difficult to check, then, that (E, f) is a $2k$-polymatroid where $f(Z) = f_1(Z) + f_2(Z)$ for all $Z \subseteq E$. We denote (E, f) by $\mathcal{Q}_1 + \mathcal{Q}_2$ or, when $\mathcal{Q}_1 = \mathcal{Q}_2$, by $2\mathcal{Q}_1$. We are not limited, however, to a sum of only 2 polymatroids. In particular, the sum of k copies of the matroid $U_{n-1,n}$, denoted $kU_{n-1,n}$, is a k-polymatroid consisting of n rank-k elements placed freely in rank $kn - k$.

Duality and minors

One attractive feature of k-polymatroids is that there are notions of duality, deletion, and contraction that mimic many of the nice properties of the same notions in matroids. Let $\mathcal{Q} = (E, f)$ be a k-polymatroid. For all subsets X of E, let

$$f^*(X) = k|X| + f(E - X) - f(E).$$

Then (E, f^*) is a k-polymatroid $\mathcal{Q}^*$, which, following [7], we call the *k-dual* of $\mathcal{Q}$.

For a subset X of E, define $f_{\mathcal{Q}\backslash X}$ and $f_{\mathcal{Q}/X}$, for all subsets A of $E - X$, by $f_{\mathcal{Q}\backslash X}(A) = f(A)$ and $f_{\mathcal{Q}/X}(A) = f(X \cup A) - f(A)$. Let $\mathcal{Q}\backslash X = (E - X, f_{\mathcal{Q}\backslash X})$ and $\mathcal{Q}/X = (E - X, f_{\mathcal{Q}/X})$. It is common to write $f\backslash X$ instead of $f_{\mathcal{Q}\backslash X}$ and f/X instead of $f_{Q/X}$. It is easy to verify that both of $\mathcal{Q}/X$ and $\mathcal{Q}\backslash X$ are k-polymatroids, and that $\mathcal{Q}^*\backslash X = (\mathcal{Q}/X)^*$. We call $\mathcal{Q}\backslash X$ and $\mathcal{Q}/X$ the *deletion* and *contraction* of X from $\mathcal{Q}$. We note that the k-dual is the unique involutory operation on the class of k-polymatroids that interchanges deletion and contraction (see [10]).

Connectivity

Following Matúš [4], we say that a k-polymatroid $\mathcal{Q} = (E, f)$ is *connected* or 2-*connected* if $f(X) + f(E - X) > f(E)$ for all nonempty proper subsets X of E; otherwise, $\mathcal{Q}$ is *disconnected*. If $f(X) + f(E - X) = f(E)$, then X is a *separator*; it is *nontrivial* if $X \notin \{\emptyset, E\}$. When X is a nontrivial separator, $(X, E - X)$ is called a 1-*separation* of $\mathcal{Q}$. It is a quick exercise to see that $\mathcal{Q}$ is connected if and only if $\mathcal{Q}^*$ is connected (see [6]). We introduce the concept of 3-connectedness for k-polymatroids in the next section.

3. Parallel connection and 2-sum

Here, we expand upon the notion of parallel connection for polymatroids that is given in [4]. This operation for polymatroids is a generalization of that for matroids in that it consists of sticking together two polymatroids as freely as possible across a designated element of each. Below, we give a formal definition that mimics the language of parallel connection for matroids.

Suppose $\mathcal{Q}_1 = (E_1, f_1)$ and $\mathcal{Q}_2 = (E_2, f_2)$ are k-polymatroids on disjoint ground sets. Let $\mathcal{Q}_1 \oplus \mathcal{Q}_2 = (E_1 \cup E_2, f)$ where $f(Z) = f_1(Z \cap E_1) + f_2(Z \cap E_2)$ for all $Z \subseteq E_1 \cup E_2$. It is well known and easily checked that $\mathcal{Q}_1 \oplus \mathcal{Q}_2$ is a k-polymatroid. Following [1], we call it the *direct sum* of $\mathcal{Q}_1$ and $\mathcal{Q}_2$. Evidently, a k-polymatroid is 2-connected if and only if it cannot be written as a direct sum of two k-polymatroids with nonempty ground sets.

Next, suppose $\mathcal{Q}_1 = (E_1, f_1)$ and $\mathcal{Q}_2 = (E_2, f_2)$ are k-polymatroids with $E_1 \cap E_2 = \{p\}$ and $f_1(p) = f_2(p)$. Let $P(\mathcal{Q}_1, \mathcal{Q}_2) = (E_1 \cup E_2, f)$ where, for all $A \subseteq E$, if $A_1 = A \cap E_1$ and $A_2 = A \cap E_2$, then

$$f(A) = \min\bigl\{f_1(A_1) + f_2(A_2), f_1(A_1 \cup p) + f_2(A_2 \cup p) - f_1(p)\bigr\}.$$

A routine check shows that $P(\mathcal{Q}_1, \mathcal{Q}_2)$ is a k-polymatroid. We call this k-polymatroid the *parallel connection* of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ with respect to the *basepoint* p. When $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are matroids, this definition of parallel connection coincides with that for matroids. A limitation of our definition of $P(\mathcal{Q}_1, \mathcal{Q}_2)$ is that it requires the basepoints to have the same rank. To rectify this, we extend the matroid operation of principal truncation (see, for example, [5, Section 7.3]).

Intuitively, the principal truncation of an element p is achieved by adding a point on p as freely as possible and then contracting the added point. To define this operation formally, let $\mathcal{Q} = (E, f)$ be a polymatroid with $p \in E$ and let f_p be the function defined, for all subsets A of E, by

$$f_p(X) = \begin{cases} f(X) - 1, & \text{if } f(X \cup p) = f(X); \\ f(X), & \text{otherwise.} \end{cases}$$

It is not difficult to check that (E, f_p) is a polymatroid. We denote it by $T_p(\mathcal{Q})$ and say that it is obtained from $\mathcal{Q}$ by *truncating* p. This operation can be repeated. For a positive integer n, we define $T_p^n(\mathcal{Q}) = T_p(T_p^{n-1}(\mathcal{Q}))$ where $T_p^0(\mathcal{Q}) = \mathcal{Q}$. It is an easy exercise to verify that $T_p^n(\mathcal{Q})$ has rank function f_p^n defined, for all $X \subseteq E$, by

$$f_p^n(X) = \begin{cases} \max\{f(X \cup p) - n, 0\}, & \text{if } f(X \cup p) - f(X) \leqslant n; \\ f(X), & \text{otherwise.} \end{cases}$$

Suppose $\mathcal{Q}_1 = (E_1, f_1)$ and $\mathcal{Q}_2 = (E_2, f_2)$ are polymatroids with $E_1 \cap E_2 = \{p\}$. Let $n = f_2(p) - f_1(p) > 0$. We expand the notion of parallel connection to this case by setting $P(\mathcal{Q}_1, \mathcal{Q}_2)$ to be $P(\mathcal{Q}_1, T_p^n(\mathcal{Q}_2))$. When $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are matroids such that p is a loop of $\mathcal{Q}_1$ and a non-loop of $\mathcal{Q}_2$, this definition coincides with that for matroids.

The following familiar properties of parallel connection hold for k-polymatroids.

Proposition 3.1. *Let $\mathcal{Q}_1 = (E_1, f_1)$ and $\mathcal{Q}_2 = (E_2, f_2)$ be polymatroids such that $E_1 \cap E_2 = \{p\}$. Then*

(i) $P(\mathcal{Q}_1, \mathcal{Q}_2)/p = \mathcal{Q}_1/p \oplus \mathcal{Q}_2/p$; *and*

(ii) *for all* $e \in E_1 - p$,

$$P(\mathcal{Q}_1, \mathcal{Q}_2)/e = P(\mathcal{Q}_1/e, \mathcal{Q}_2) \quad \textit{and} \quad P(\mathcal{Q}_1, \mathcal{Q}_2)\backslash e = P(\mathcal{Q}_1\backslash e, \mathcal{Q}_2).$$

Proof. The proof of this proposition is not significantly different from the proof of the corresponding result for matroids (see, for example, [5]) and is omitted. □

The following result of Oxley and Whittle (see [6, Theorem 3.1]) is used throughout the paper.

Lemma 3.2. *Let* $\mathcal{Q} = (E, f)$ *be a connected* k*-polymatroid where* $|E| \geqslant 2$ *and let* A *be a nonempty proper subset of* E*. If*

$$f(A) + f(E - A) - f(E) < \min\bigl\{f(X) + f(E - X) - f(E)\colon \emptyset \neq X \subsetneq E\bigr\},$$

then at least one of $\mathcal{Q}/A$ *and* $\mathcal{Q}\backslash A$ *is connected.* □

From this lemma, we obtain the following result on non-essential elements. Recall that an element e of a connected k-polymatroid $\mathcal{Q}$ is non-essential if either $\mathcal{Q}\backslash e$ or $\mathcal{Q}/e$ is connected.

Proposition 3.3. *If* $\mathcal{Q} = (E, f)$ *is a connected* k*-polymatroid and* $e \in E$ *such that* $f(e) = 1$*, then* e *is non-essential.*

Proof. This is an immediate consequence of Lemma 3.2. □

Theorem 3.4. *Suppose* $\mathcal{Q}_1 = (E_1, f_1)$ *and* $\mathcal{Q}_2 = (E_2, f_2)$ *are* k*-polymatroids such that* $E_1 \cap E_2 = \{p\}$ *where* $f_1(p) = f_2(p)$*. Then both* $\mathcal{Q}_1$ *and* $\mathcal{Q}_2$ *are connected if and only if* $P(\mathcal{Q}_1, \mathcal{Q}_2)$ *is connected. Further, if* $P(\mathcal{Q}_1, \mathcal{Q}_2)\backslash p$ *is connected, then* $P(\mathcal{Q}_1, \mathcal{Q}_2)$ *is connected.*

Proof. If $(X, Y \cup p)$ is a 1-separation of $\mathcal{Q}_1$, then it is not difficult to check that $(X, E_2 \cup Y)$ is a 1-separation of $P(\mathcal{Q}_1, \mathcal{Q}_2)$ and that $(X, (E_2 - p) \cup Y)$ is a 1-separation of $P(\mathcal{Q}_1, \mathcal{Q}_2)\backslash p$. On the other hand, suppose $(X, Y \cup p)$ is a 1-separation of $P(\mathcal{Q}_1, \mathcal{Q}_2)$, and f_3 is the rank function for $P(\mathcal{Q}_1, \mathcal{Q}_2)$. Let $X_i = X \cap E_i$ and $Y_i = Y \cap E_i$ for each $i \in \{1, 2\}$, and observe that

$$f_3(X) = \min\bigl\{f_1(X_1) + f_2(X_2), f_1(X_1 \cup p) + f_2(X_2 \cup p) - f_1(p)\bigr\};$$
$$f_3(Y \cup p) = f_1(Y_1 \cup p) + f_2(Y_2 \cup p) - f_1(p); \quad \text{and}$$
$$f_3(E_1 \cup E_2) = f_1(E_1) + f_2(E_2) - f_1(p).$$

If $f_1(X_1) + f_2(X_2) \leqslant f_1(X_1 \cup p) + f_2(X_2 \cup p) - f_1(p)$, then since $f_3(X) + f_3(Y \cup p) = f_3(E_1 \cup E_2)$, we have

$$f_1(X_1) + f_2(X_2) + f_1(Y_1 \cup p) + f_2(Y_2 \cup p) = f_1(E_1) + f_2(E_2).$$

As $f_i(X_i) + f_i(Y_i \cup p) \geqslant f_i(E_i)$ for each $i \in \{1, 2\}$, it follows that $(X_i, Y_i \cup p)$ is a 1-separation for each $i \in \{1, 2\}$. On the other hand, if $f_1(X_1) + f_2(X_2) > f_1(X_1 \cup p) + f_2(X_2 \cup p) - f_1(p)$, then, as $f_3(X) + f_3(Y \cup p) = f_3(E_1 \cup E_2)$, we have

$$f_1(X_1 \cup p) + f_2(X_2 \cup p) + f_1(Y_1 \cup p) + f_2(Y_2 \cup p) - f_1(p) = f_1(E_1) + f_2(E_2).$$

From submodularity again, it follows that $f_2(p) = f_1(p) = 0$, and thus $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are disconnected. □

In addition to parallel connection, we make use of the 2-sum operation. Let $\mathcal{Q}_1$ and $\mathcal{Q}_2$ be k-polymatroids on ground sets E_1 and E_2, respectively, with $E_1 \cap E_2 = \{p\}$. If $f_1(p) = f_2(p) = 1$ and p is not a separator for either $\mathcal{Q}_1$ or $\mathcal{Q}_2$, then the 2-*sum* of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ is defined to be $P(\mathcal{Q}_1, \mathcal{Q}_2)\backslash p$ and denoted $\mathcal{Q}_1 \oplus_2 \mathcal{Q}_2$. The following shows some fundamental connectivity properties of this 2-sum operation.

Corollary 3.5. *Suppose $\mathcal{Q}_1 = (E_1, f_1)$ and $\mathcal{Q}_2 = (E_2, f_2)$ are k-polymatroids such that $E_1 \cap E_2 = \{p\}$ where $f_1(p) = f_2(p) = 1$. Then the following are equivalent.*

(i) *$\mathcal{Q}_1$ and $\mathcal{Q}_2$ are both connected*;
(ii) *$\mathcal{Q}_1 \oplus_2 \mathcal{Q}_2$ is connected*;
(iii) *$P(\mathcal{Q}_1, \mathcal{Q}_2)$ is connected.*

Proof. Using Theorem 3.4, we have only to show that (iii) implies (ii). From Proposition 3.1, we observe that $P(\mathcal{Q}_1, \mathcal{Q}_2)/p$ is disconnected. Since $f_1(p) = f_2(p) = 1$, we use Proposition 3.3 to see that p is non-essential and therefore that $P(\mathcal{Q}_1, \mathcal{Q}_2)\backslash p = \mathcal{Q}_1 \oplus_2 \mathcal{Q}_2$ is connected. $\square$

We say that a k-polymatroid $\mathcal{Q}$ is 3-*connected* if and only if it cannot be written as a 2-sum of a pair of k-polymatroids each with fewer elements than $\mathcal{Q}$. The following proposition allows us to give an alternative definition.

Proposition 3.6. *Suppose $\mathcal{Q} = (E, f)$ is a k-polymatroid for which there exists a partition (X_1, X_2) of E such that $f(X_1) + f(X_2) = f(E) + 1$ and $\min\{|X_1|, |X_2|\} \geqslant 2$. Then there are polymatroids $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on ground sets $X_1 \cup p$ and $X_2 \cup p$, respectively, where p is a new point not in E, such that $\mathcal{Q} = \mathcal{Q}_1 \oplus_2 \mathcal{Q}_2$.*

Proof. For $(i, j) \in \{(1, 2), (2, 1)\}$, let $\mathcal{Q}_i = (X_i \cup p, f_i)$ where f_i is defined, for all $A \subseteq X_i \cup p$, by

$$f_i(A) = \begin{cases} f((A-p) \cup X_j) - f(X_j) + 1 & \text{if } p \in A; \\ f(A) & \text{if } p \notin A. \end{cases}$$

It is routine to check that f_i is a k-polymatroid. Let f_3 be the rank function of $P(\mathcal{Q}_1, \mathcal{Q}_2)$. Since $\mathcal{Q}_1 \oplus_2 \mathcal{Q}_2 = P(\mathcal{Q}_1, \mathcal{Q}_2)\backslash p$, it suffices to show that $f_3(A) = f(A)$ for all subsets A of E. Choose such a subset A, let $A_i = A \cap X_i$ for $i \in \{1, 2\}$, and note that

$$\begin{aligned} f_3(A) &= \min\bigl\{f_1(A_1) + f_2(A_2),\, f_1(A_1 \cup p) + f_2(A_2 \cup p) - f_1(p)\bigr\} \\ &= \min\bigl\{f(A_1) + f(A_2),\, f(A_1 \cup X_2) + f(A_2 \cup X_1) - f(E)\bigr\}. \end{aligned}$$

Observe that if U and V are disjoint subsets of E with $S \subseteq U$ and $T \subseteq V$, then

$$\begin{aligned} f(U) + f(V) + f(S \cup T) &\geqslant f(U) + f(S \cup V) + f(T) \\ &\geqslant f(U \cup V) + f(S) + f(T). \end{aligned}$$

Rearranging this inequality provides that

$$f(U) + f(V) - f(U \cup V) \geqslant f(S) + f(T) - f(S \cup T). \tag{3.1}$$

Since $f(X_1) + f(X_2) = f(E) + 1$, we have from (3.1) that

$$f(A_1 \cup X_2) \in \bigl\{f(A_1) + f(X_2),\, f(A_1) + f(X_2) - 1\bigr\},$$

with $f(A_2 \cup X_1)$ behaving similarly. If $f(A_1 \cup X_2) = f(A_1) + f(X_2)$, then another application of (3.1) shows that

$$f(A_1) + f(A_2) = f(A_1 \cup A_2).$$

From submodularity, we have that $f(A_1 \cup X_2) + f(A_2 \cup X_1) - f(E) \geqslant f(A_1 \cup A_2)$, and it follows that $f_3(A) = f(A_1) + f(A_2) = f(A)$, as desired. By symmetry, then, we have only to consider when $f(A_1 \cup X_2) = f(A_1) + f(X_2) - 1$ and $f(A_2 \cup X_1) = f(A_2) + f(X_1) - 1$. In this case, we observe that

$$\begin{aligned} f(A_1) + f(A_2) &= f(A_1 \cup X_2) + f(X_1 \cup A_2) - f(X_1) - f(X_2) + 2 \\ &= f(A_1 \cup X_2) + f(X_1 \cup A_2) - f(E) + 1 \\ &\geqslant f(E) + f(A) - f(E) + 1 \\ &= f(A) + 1. \end{aligned}$$

From this, it follows that

$$f_3(A) = f(A_1 \cup X_2) + f(X_1 \cup A_2) - f(E) = f(A_1) + f(A_2) - 1,$$

and with an application of (3.1), that

$$f(A_1) + f(A_2) = f(A) + 1.$$

Combining these equations yields that $f_3(A) = f(A)$ and the conclusion holds. □

Corollary 3.7. *A k-polymatroid $\mathcal{Q} = (E, f)$ is 3-connected if and only if for any partition (X, Y) of E with $f(X) + f(Y) = f(E) + 1$, either $|X| = 1$ or $|Y| = 1$.*

From this, it is clear that a k-polymatroid $\mathcal{Q}$ is 3-connected if and only if $\mathcal{Q}^*$ is 3-connected. Our final result shows that 2-summing commutes for k-polymatroids. We omit the proof since it involves a routine, but tedious, exhaustive case-check.

Proposition 3.8. *For $i \in \{1, 2, 3\}$, let $\mathcal{Q}_i = (E_i, f_i)$ be a k-polymatroid for which $E_1 \cap E_2 = \{p_1\}$ and $E_2 \cap E_3 = \{p_2\}$ with $f_1(p_1) = f_2(p_1) = f_2(p_2) = f_3(p_2) = 1$. Then $\mathcal{Q}_1 \oplus_2 (\mathcal{Q}_2 \oplus_2 \mathcal{Q}_3) = (\mathcal{Q}_1 \oplus_2 \mathcal{Q}_2) \oplus_2 \mathcal{Q}_3$.*

4. Non-essential elements

Recall that an element e of a connected k-polymatroid $\mathcal{Q}$ is non-essential if either $\mathcal{Q} \backslash e$ or $\mathcal{Q}/e$ is connected. Tutte showed in [9] that every element of a connected matroid is non-essential. We expand this result to k-polymatroids by determining the number of non-essential elements that are guaranteed to exist in any k-polymatroid. To do so, we make extensive use of the truncation operation defined in the previous section.

Lemma 4.1. *Let $\mathcal{Q} = (E, f)$ be a connected k-polymatroid with $e \in E$. Then $T_e(\mathcal{Q})$ is connected if and only if $\mathcal{Q}$ is connected with $f(e) > 1$.*

Proof. Let (A, B) be a partition of E with $e \in A$ and B nonempty. Suppose $T_e(\mathcal{Q})$ is connected. Then certainly $f(e) > 1$ or else e would be a loop in $T_e(\mathcal{Q})$. To show that $\mathcal{Q}$ is connected, observe that

$$\begin{aligned} f(A) + f(B) &= f_e(A) + f(B) + 1 \\ &\geqslant f_e(A) + f_e(B) + 1 \\ &> f_e(E) + 1 \\ &= f(E). \end{aligned}$$

We now assume that $\mathcal{Q}$ is connected with $f(e) > 1$. Then

$$f_e(A) + f_e(B) \geqslant f(A) + f(B) - 2 \geqslant f(E) - 1 = f_e(E),$$

and it thus suffices to consider the case when both $f_e(B) = f(B) - 1$ and $f(A) + f(B) = f(E) + 1$. From the first of these equations, we have $f(B \cup e) = f(B)$ and so, from the second equation, get $f(A) + f(B \cup e) = f(E) + 1$. It follows from submodularity that

$$f(E) + f(e) \leqslant f(A) + f(B \cup e) = f(E) + 1,$$

and therefore $f(e) \leqslant 1$. $\square$

Lemma 4.2. *Let $\mathcal{Q} = (E, f)$ be a k-polymatroid with $e \in E$ and disjoint sets $C, D \subseteq E - e$ such that $f(C \cup e) > f(C)$. Then $T_e(\mathcal{Q})\backslash D/C = T_e(\mathcal{Q}\backslash D/C)$.*

Proof. Let $X \subseteq E - (C \cup D)$. It is straightforward to show that $f_e\backslash D/C(X) = (f\backslash D/C)_e(X)$. $\square$

Theorem 4.3. *Every connected k-polymatroid having at least two elements has at least two non-essential elements.*

Proof. Let $\mathcal{Q} = (E, f)$ be a connected k-polymatroid with $|E| \geqslant 2$. We proceed by induction on the rank of $\mathcal{Q}$. If $f(E) = 0$, then $\mathcal{Q}$ is not connected and we are done. Thus we assume the theorem holds for polymatroids of rank less than that of $\mathcal{Q}$. If possible, choose $e \in E$ such that $f(E - e) < f(E)$. If each $e \in E$ satisfies $f(E - e) = f(E)$, then choose $e \in E$ such that $f(e) = \max\{f(x)\colon x \in E\}$. If $f(e) = 1$, then $\mathcal{Q}$ consists entirely of rank-1 elements and so consists entirely of non-essential elements by Proposition 3.3. Otherwise, we use Lemma 4.1 to see that $T_e(\mathcal{Q})$ is a connected k-polymatroid with at least two elements and rank one less than the rank of $\mathcal{Q}$. By induction, then, we may pick two elements $a, b \in E$ that are non-essential in $T_e(\mathcal{Q})$. By combining Lemmas 4.1 and 4.2, we note that if an element of $E - e$ is non-essential in $T_e(\mathcal{Q})$, then it is non-essential in $\mathcal{Q}$. Therefore we need only show that either e is non-essential in $\mathcal{Q}$, or there are two elements $x, y \in E - e$ that are non-essential in $T_e(\mathcal{Q})$. Clearly, if $a, b \in E - e$, then we are done. Thus assume that e is non-essential in $T_e(\mathcal{Q})$. If $f(E - e) < f(E)$, then it is not difficult to show that e is essential in $\mathcal{Q}$. Hence assume that $f(E - x) = f(E)$ for all $x \in E$. If $T_e(\mathcal{Q})/e$ is connected, then, as $T_e(\mathcal{Q})/e = \mathcal{Q}/e$, the theorem holds. Hence we may assume that $T_e(\mathcal{Q})\backslash e$ is connected. If $|E| = 2$, then the result is obvious and so $T_e(\mathcal{Q})\backslash e$ is a connected k-polymatroid with at least two elements. Let x and y be non-essential in $T_e(\mathcal{Q})\backslash e$.

If $T_e(\mathcal{Q})\backslash\{e, x\}$ is connected, then $T_e(\mathcal{Q})\backslash\{x\}$ is connected unless

$$f_e(e) + f_e\big(E - \{e, x\}\big) = f_e(E - x) = f_e(E). \tag{4.1}$$

In this case, suppose $(A \cup x, B)$ partitions $E - e$ nontrivially such that

$$f/e(A \cup x) + f/e(B) = f/e(E - e).$$

Then

$$f\big(A \cup \{e, x\}\big) + f(B \cup e) = f(E) + f(e). \tag{4.2}$$

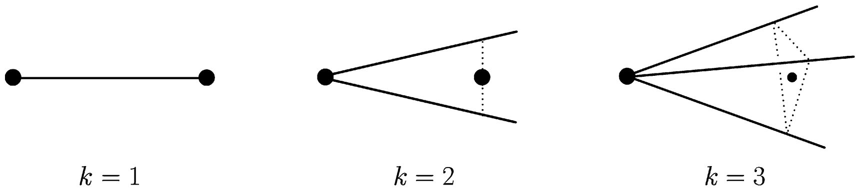

Fig. 1. $\mathcal{S}_k$ for $k \in \{1,2,3\}$.

Observe, however, that (4.1) implies that $f(e)+f(E-\{e,x\})=f(E)$ and thus, since $B \subseteq E-\{e,x\}$, that $f(e)+f(B)=f(B\cup e)$. Applying this to (4.2) shows that $(A\cup\{e,x\},B)$ is a 1-separation of $\mathcal{Q}$, a contradiction. It remains to consider the case when $T_e(\mathcal{Q})\backslash e/x$ is connected. By a similar argument to the above, we have that $(e,E-x)$ is the only possible 1-separation of $T_e(\mathcal{Q})/x$. If $\mathcal{Q}/x$ is connected, we are done. Thus assume that $(A\cup e,B)$ is a 1-separation of $\mathcal{Q}/x$. Since $(f/x)_e(A\cup e)=f/x(A\cup e)-1$ and $(f/x)_e(E-x)=f/x(E-x)-1$, it follows that

$$(f/x)_e(A\cup e)+f/x(B)=(f/x)_e(E-x). \tag{4.3}$$

Now, either $f/x(B)=(f/x)_e(B)$ or $f/x(B)=(f/x)_e(B)+1$. Observe that if $f(\{x,e\})=f(x)$, then $\mathcal{Q}\backslash e$ is connected and we are done. Thus $f(\{x,e\})>f(x)$ and we have, from Lemma 4.2, that $T_e(\mathcal{Q}/x)=T_e(\mathcal{Q})/x$. Thus if $f/x(B)=(f/x)_e(B)+1$, then (4.3) becomes

$$f_e/x(A\cup e)+f_e/x(B)=f_e/x(E-x)-1,$$

contradicting the submodularity of $T_e(\mathcal{Q})/x$. On the other hand, if $f/x(B)=f_e/x(B)$, then

$$f_e/x(A\cup e)+f_e/x(B)=f_e/x(E-x).$$

As $(e,E-x)$ is the only possible 1-separation of $T_e(\mathcal{Q})/x$, it follows that $A=\emptyset$. Then, since $(A\cup e,B)$ is a 1-separation of $\mathcal{Q}/x$,

$$f/x(e)+f/x\big(E-\{e,x\}\big)=f/x(E-x).$$

Since $f/x(E-\{e,x\})=f/x(E-x)$, it follows that $f(\{x,e\})=f(e)$ and thus $\mathcal{Q}\backslash x$ is connected. $\square$

We now know that every connected k-polymatroid has at least two non-essential elements. The next example shows that this bound is sharp.

Example 4.4. Choose integers $k\geqslant 1$ and $n\geqslant 1$. Let E be a set with $|E|=k$ and choose distinct elements $a,b\notin E$. Take $M=(E\cup\{a,b\},r)$ to be a matroid isomorphic to $U_{1,k+1}\oplus U_{0,1}$ where b is the loop and $\mathcal{Q}=(E\cup\{a,b\},f)$ to be an n-polymatroid isomorphic to $nU_{k,k+1}\oplus U_{0,1}$ where a is the loop. Then the $(n+1)$-polymatroid $M+\mathcal{Q}$ has a and b as its only non-essential elements. If $n=1$, then we denote $M+\mathcal{Q}$ by $\mathcal{S}_k$ for each k. The 2-polymatroid $\mathcal{S}_k$ is shown geometrically in Fig. 1 for $k\in\{1,2,3\}$.

Lemma 4.5. *If $\mathcal{Q}=(E,f)$ is a connected 2-polymatroid and, for some $e\in E$, both $f\backslash e$ and f/e are not connected, then $f(E)=f(E-e)$, $f(e)=2$, and $f(X)+f(E-X)-f(E)=1$ for some set $X\subsetneq E$.*

Proof. This is an immediate consequence of Lemma 3.2. $\square$

If $\mathcal{Q}=(E,f)$ is a 2-polymatroid and $x\in E$ such that $f(E-x)=f(E)-1$, then $f^*(e)=1$ and we say that e is a *copoint*. In the following theorem, we show that the polymatroids given in Example 4.4

when $n = 1$ are the only 3-connected 2-polymatroids with exactly 2 non-essential elements and no copoints. After obtaining this result, it is not difficult to remove the no-copoints requirement, which is done in Corollary 4.7.

Theorem 4.6. *If $\mathcal{Q}$ is a 3-connected 2-polymatroid with at least three elements, no copoints, and exactly two non-essential elements, then $\mathcal{Q}$ is isomorphic to $\mathcal{S}_k$ for some k.*

Proof. Let $\mathcal{Q} = (E, f)$ be a 3-connected 2-polymatroid with an essential element a. Choose a nontrivial partition (X, Y) of $E - a$ with $|X|$ maximal such that $f(X \cup a) + f(Y \cup a) = f(E) + 2$. A partition of this type is a 1-separation of $\mathcal{Q}/a$ and thus exists. Similarly, choose a partition (A, B) of $E - a$ with $|A|$ maximal such that $f(A) + f(B) = f(E)$. Then

$$\begin{aligned} 2f(E) + 2 &= f(A) + f(B) + f(X \cup a) + f(Y \cup a) \\ &\geqslant f(A \cup X \cup a) + f(B \cap Y) + f(B \cup Y \cup a) + f(A \cap X) \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} 2f(E) + 2 &= f(A) + f(B) + f(X \cup a) + f(Y \cup a) \\ &\geqslant f(A \cup Y \cup a) + f(B \cap X) + f(B \cup X \cup a) + f(A \cap Y). \end{aligned} \tag{4.5}$$

Since $\mathcal{Q}$ is 3-connected, we get from (4.4) that at least one of $|A \cap X|$ and $|B \cap Y|$ is less than 2. In fact, for some $k \geqslant 1$,

$$\big(|A \cap X|, |B \cap Y|\big) \in \big\{(0, k), (0, 0), (1, 1), (k, 0)\big\}.$$

If $|A \cap X| = 0$ and $|B \cap Y|$ is nonzero, then (4.5) tells us that, since neither $A \cap Y$ nor $B \cap X$ may be empty, both must be singletons. Thus $A = (X \cap A) \cup (Y \cap A) = A \cap Y$ and A is a singleton. However, B then contains at least two elements, contradicting the maximality of A.

Next, we assume both $A \cap X$ and $B \cap Y$ are empty. Again, from (4.5), we get that $|A \cap Y| = |B \cap X| = 1$. Let $x \in B \cap X$ and $y \in A \cap Y$. Since $f(A \cup Y \cup a) + f(B \cap X) = f(E) + 1$, we have that $f(\{a, y\}) + f(x) = f(E) + 1$. As $\mathcal{Q}$ has no copoints, it follows that $f(x) = 1$ and similarly that $f(y) = 1$. It follows, since $f(A) + f(B) = f(E)$, that $f(E) = 2$, so $f(\{a, y\}) = f(\{a, x\}) = 2$. If $f(\{x, y\}) = 1$, then $f(\{x, y\}) + f(a) = f(E) + 1$, which is impossible since $f(a) = 2$. Therefore, $\mathcal{Q} \cong \mathcal{S}_1$.

Now, we assume $|A \cap X| = |B \cap Y| = 1$ and let $A \cap X = \{x\}$; $B \cap Y = \{y\}$. From (4.5) and the maximality of A and X, we have $|A \cap Y| = |B \cap X| \leqslant 1$. If $|A \cap Y| = |B \cap X| = 0$, then, similarly to the previous case, we have that $\mathcal{Q} \cong \mathcal{S}_1$. We thus assume $|A \cap Y| = |B \cap X| = 1$ and let $\{w\} = B \cap X$ and $\{z\} = A \cap Y$. From this, we may use (4.4) and (4.5) to get $f(w) = f(z) = f(x) = f(y) = 1$. As rank-1 elements are always non-essential, this contradicts that $\mathcal{Q}$ has exactly two non-essential elements.

Finally, we consider the case when $|B \cap Y| = 0$ and $A \cap X$ is nonempty. Arguing as above, we find that each of $B \cap X$ and $A \cap Y$ consists of a single rank-one element, which we call x and y, respectively. Using (4.4), (4.5), and the 3-connectedness of $\mathcal{Q}$, we are able to find that $f(\{a, y\}) = 2$, $f(\{a, x, y\}) = 3$, $f(E - \{a, x\}) = f(E) - 1$, $f(E - \{a, y\}) = f(E)$, $f(E - \{a, x, y\}) = f(E) - 1$, and $f(\{a, x\}) = 3$. Indeed, as x and y are points, they are the sole non-essential elements of $\mathcal{Q}$. Thus we may choose $b \in E - \{a, x, y\}$ and note that b must be essential. If we repeat the previous steps of this proof using b instead of a, we come to the conclusion that b satisfies $f(\{b, y\}) = 2$, $f(\{b, x, y\}) = 3$, $f(E - \{b, x\}) = f(E) - 1$, $f(E - \{b, y\}) = f(E)$, $f(E - \{b, x, y\}) = f(E) - 1$, and $f(\{b, x\}) = 3$. As b was chosen arbitrarily, we have that these equations are satisfied for all $p \in E - \{x, y\}$.

Since, for each $p \in E - \{x, y\}$, we have that $f(\{p, y\}) = 2$, it follows that $f(E - x) \leqslant |E| - 1$ and thus $f(E) \leqslant |E| - 1$. If possible, choose a minimal set $P \subseteq E - \{x, y\}$ for which $f(P) \leqslant |P|$ and let $b \in P$. By the minimality of P, we have $f(P - b) \geqslant |P - b| + 1 = |P| \geqslant f(P)$ and thus $f(P - b) = f(P)$. Recall, however, that $f(x) + f(E - \{b, x\}) = f(E)$. Since $P - b \subseteq E - \{b, x\}$, it follows that $f(E - x) =$

$f(E - \{b, x\})$, contradicting the connectivity of $\mathcal{Q}$. Therefore, for all $L \subseteq E - \{x, y\}$, we have $f(L) = |L| + 1$ and thus $f(E) = |E| - 1$. It then follows that $\mathcal{Q} \cong \mathcal{S}_{|E|-2}$, as desired. $\square$

In a connected 2-polymatroid $\mathcal{Q} = (E, f)$, if $x \in E$ has $f(x) = 1$, then we may turn x into a copoint by defining $\mathcal{Q}^e = (E, f^e)$ where, for all $X \subseteq E$, we have

$$f^e(X) = \begin{cases} f(X) + 1 & \text{if } e \in X; \\ f(X) & \text{otherwise.} \end{cases}$$

We call this operation *element expansion*.

Corollary 4.7. *Every* 3*-connected* 2*-polymatroid on at least three elements with exactly two non-essential elements can be obtained from some $\mathcal{S}_n$ by performing a sequence of element expansions.*

Proof. Suppose $\mathcal{Q} = (E, f)$ is such a 2-polymatroid having $\{x_1, x_2, \dots, x_n\}$ as its set of copoints. Let $\mathcal{R} = T_{x_1}(T_{x_2}(\cdots T_{x_n}(\mathcal{Q}))\cdots)$. It is not difficult to check that $\mathcal{R}$ is 3-connected and we can use Lemmas 4.1 and 4.2 to see that $\mathcal{R}$ has exactly two non-essential elements. From Theorem 4.6, we have that $\mathcal{R}$ is isomorphic to $\mathcal{S}_n$ for some n. The conclusion follows. $\square$

We conclude by characterizing all those 2-polymatroids with exactly two non-essential elements. The following proposition will be helpful to this end.

Proposition 4.8. *Let $\mathcal{Q}_1 = (E_1, f_1)$ and $\mathcal{Q}_2 = (E_2, f_2)$ be connected k-polymatroids such that $E_1 \cap E_2 = \{p\}$ and $f_1(p) = f_2(p) = 1$. An element x in $(E_1 \cup E_2) - p$ is non-essential in either $\mathcal{Q}_1$ or $\mathcal{Q}_2$ if and only if x is non-essential in $\mathcal{Q}_1 \oplus_2 \mathcal{Q}_2$.*

Proof. From Proposition 3.1,

$$(\mathcal{Q}_1 \oplus_2 \mathcal{Q}_2)\backslash x = P(\mathcal{Q}_1, \mathcal{Q}_2)\backslash\{x, p\} = P(\mathcal{Q}_1\backslash x, \mathcal{Q}_2)\backslash p = (\mathcal{Q}_1\backslash x) \oplus_2 \mathcal{Q}_2.$$

Similarly, $(\mathcal{Q}_1 \oplus_2 \mathcal{Q}_2)/x = (\mathcal{Q}_1/x) \oplus_2 \mathcal{Q}_2$. By combining these equations with Corollary 3.5, we obtain the proposition. $\square$

The connected 2-polymatroids with exactly two non-essential elements consist of the members of $\{\mathcal{S}_1, \mathcal{S}_2, \dots\}$ along with paths of 2-sums of such 2-polymatroids where the basepoints of the 2-sums are non-essential in both summands.

Theorem 4.9. *Let $\mathcal{Q}$ be a connected* 2*-polymatroid with at least three elements. Then $\mathcal{Q}$ has exactly two non-essential elements if and only if, for some $n \geqslant 1$, there is a sequence $\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_n$ of* 2*-polymatroids such that*

(i) *each $\mathcal{Q}_i$ is isomorphic to some member of $\{U_{1,2} + U_{1,1}, \mathcal{S}_1, \mathcal{S}_2, \dots\}$;*
(ii) *if either $n = 1$ or $2 \leqslant i \leqslant n - 1$, then $\mathcal{Q}_i$ is isomorphic to some member of $\{\mathcal{S}_1, \mathcal{S}_2, \dots\}$;*
(iii) *the ground sets of $\mathcal{Q}_1, \mathcal{Q}_2, \dots, \mathcal{Q}_n$ are disjoint except that, for each i in $\{1, 2, \dots, n - 1\}$, the sets $E(\mathcal{Q}_i)$ and $E(\mathcal{Q}_{i+1})$ meet in a single rank-*1 *element; and*
(iv) $\mathcal{Q} \cong \mathcal{Q}_1 \oplus_2 \mathcal{Q}_2 \oplus_2 \cdots \oplus_2 \mathcal{Q}_n$.

Proof. If we have a sequence satisfying the four conditions, Proposition 4.8 implies that $\mathcal{Q}$ has exactly two non-essential elements. For the converse, we proceed by induction on the rank of E. If $f(E) = 1$, then, since $|E| > 2$ and $\mathcal{Q}$ is connected, it follows that $\mathcal{Q}$ consists of $|E|$ points, each of which must be non-essential, a contradiction. Thus assume $f(E) > 1$ and that the conclusion holds for 2-polymatroids of rank less than $f(E)$. If $\mathcal{Q}$ is 3-connected, then, from Corollary 4.7, there are three possibilities:

$n=1$ with $\mathcal{Q}_1$ isomorphic to some member of $\{\mathcal{S}_1, \mathcal{S}_2, \ldots\}$; $n=2$ with $\mathcal{Q}_1$ isomorphic to $U_{1,2}+U_{1,1}$ and $\mathcal{Q}_2$ isomorphic to some member of $\{\mathcal{S}_1, \mathcal{S}_2, \ldots\}$; or $n=3$ with both $\mathcal{Q}_1$ and $\mathcal{Q}_3$ isomorphic to $U_{1,2}+U_{1,1}$ and $\mathcal{Q}_2$ isomorphic to some member of $\{\mathcal{S}_1, \mathcal{S}_2, \ldots\}$. We thus assume that $\mathcal{Q}$ is not 3-connected. Choose a nontrivial partition (X, Y) of E such that $f(X)+f(Y)=f(E)+1$ and $2 \leqslant |X| \leqslant |Y|$. If $f(X)=1$, then each member of X is a point and is thus non-essential. As $\mathcal{Q}$ has exactly two non-essential elements, X consists of two points which are necessarily parallel. However, $\mathcal{Q} \backslash x$, where $x \in X$, is connected with two non-essential elements. Clearly these two non-essential elements are also non-essential in $\mathcal{Q}$, a contradiction. Therefore $f(X)>1$ and thus $f(Y)<f(E)$. Similarly, $f(X)<f(E)$. We now use Proposition 3.6 to choose 2-polymatroids $\mathcal{Q}_1$ and $\mathcal{Q}_2$ on ground sets $X \cup p$ and $Y \cup p$, respectively, where p is a point not in E and $\mathcal{Q}=\mathcal{Q}_1 \oplus_2 \mathcal{Q}_2$. Moreover, the ranks of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are each less than that of $\mathcal{Q}$. If x is a non-essential element of $\mathcal{Q}_1$ that meets E, then, by using Proposition 4.8, x is a non-essential element of $\mathcal{Q}$. Thus each of $\mathcal{Q}_1$ and $\mathcal{Q}_2$ has p as a non-essential element and has exactly one other non-essential element. By induction, $\mathcal{Q}_1$ and $\mathcal{Q}_2$ satisfy the four conditions in the theorem. It follows immediately that the 2-sum of $\mathcal{Q}_1$ and $\mathcal{Q}_2$, that is $\mathcal{Q}$, satisfies the four conditions. $\square$

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